

GROUND STATES FOR MEAN FIELD MODELS WITH A TRANSVERSE COMPONENT

DMITRY IOFFE AND ANNA LEVIT

ABSTRACT. We investigate global logarithmic asymptotics of ground states for a family of quantum mean field models. Our approach is based on a stochastic representation and a combination of large deviation and weak KAM techniques. The spin- $\frac{1}{2}$ case is worked out in more detail.

1. THE MODEL AND THE RESULT

1.1. Introduction. Stochastic representations/path integral approach frequently provides a useful intuition and insight into the structure of quantum spin states. Numerous examples include [2, 3, 8, 10, 12, 17, 18, 22, 26]. In this work we rely on a path integral approach and related large deviations techniques, and derive *global* logarithmic asymptotics of ground states for a class of quantum mean field models in transverse field. These asymptotics limits are identified as weak-KAM [16] type solutions of certain Hamilton-Jacobi equations. In principle, such solutions are not unique, and an additional refined analysis along the lines of [14, 19, 20] is needed for recovering the correct asymptotic ground state. This issue is addressed in more detail for the spin $\frac{1}{2}$ -case. In particular, our results imply logarithmic asymptotics of ground states for models with p -body interactions [5].

In the case of Laplacian with periodic potential a weak KAM approach to semi-classical asymptotics was already employed in [1].

Our stochastic representation gives rise to a family of continuous time Markov chains on a simplex Δ_d^N (defined below) of $\frac{1}{N}\mathbb{Z}^d$. The transition rates are enhanced by a factor of N , and the chain moves in a potential of the type NF . Ground states are Perron-Frobenius eigenfunctions of the corresponding generators. On the concluding stages of this work we have learned about the series of papers [23–25]. The models we consider here essentially fall into a much more general framework studied in these works. The authors of [23–25] extend an analysis of Schrödinger operators [14, 19, 20] on \mathbb{R}^d to lattice operators on $\epsilon\mathbb{Z}^d$, and they develop powerful techniques, which go well beyond the scope of our work, and which enable a complete asymptotic expansion of low lying eigenvalues and eigenfunctions in neighbourhoods of potential wells.

The paper is organized as follows: The class of models is described in Subsection 1.2, and the results are formulated in Subsection 1.4. Main steps of our approach are explained in Section 2, whereas some of the proofs are relegated to Section 3. The

spin- $\frac{1}{2}$ case is studied in Section 4. Finally, in the Appendix, we establish the required properties of the Lagrangian \mathcal{L}_0 in (1.11) and, accordingly, the required regularity properties of local minimizers.

1.2. Class of Models. Let \mathbb{X} be a d -dimensional complex Hilbert space. For the rest of the paper we fix an orthonormal basis $\{|\alpha\rangle\}_{\alpha \in \mathcal{A}}$ of \mathbb{X} . We refer to the set \mathcal{A} of cardinality d as the set of classical labels. Denote projections $P_\alpha \triangleq |\alpha\rangle\langle\alpha|$. The induced basis of $\mathbb{X}_N = \otimes_1^N \mathbb{X}$ is

$$|\underline{\alpha}\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_N\rangle \quad \alpha_1, \dots, \alpha_N \in \mathcal{A}.$$

The corresponding lifting of the projection operator acting on i -th component is $P_\alpha^i = I \otimes \cdots \otimes I \otimes P_\alpha \otimes I \otimes \cdots \otimes I$. For $\alpha \in \mathcal{A}$ set $M_\alpha^N = \frac{1}{N} \sum_i P_\alpha^i$. Let \underline{M}^N be the d -dimensional vector with operator entries M_α^N .

We are ready to define the Hamiltonian \mathcal{H}_N which acts on \mathbb{X}_N ,

$$-\mathcal{H}_N = NF(\underline{M}^N) + \sum_i B_i. \quad (1.1)$$

Above, B_i -s are copies of a Hermitian matrix B on \mathbb{X} , B_i acts on the i -th component of $|\underline{\alpha}\rangle$.

We assume:

A1. F is a real polynomial of finite degree.

Let Δ_d be the simplex, $\Delta_d = \{\underline{m} \in \mathbb{R}_+^d : \sum m_i = 1\}$. In the sequel we shall write $\text{int}(\Delta_d)$ for the relative interior of Δ_d . Accordingly, $\partial\Delta_d \triangleq \Delta_d \setminus \text{int}(\Delta_d)$,

Given $\underline{m} \in \Delta_d$ and a basis vector $\underline{\alpha} \in \mathcal{A}^N$ let us say that $\underline{\alpha} \sim \underline{m}$, or, equivalently, $\underline{m} = \underline{m}(\underline{\alpha})$, if

$$m_\alpha = \frac{\#\{i : \alpha_i = \alpha\}}{N} \Leftrightarrow M_\alpha^N |\underline{\alpha}\rangle = m_\alpha |\underline{\alpha}\rangle, \quad (1.2)$$

for all $\alpha \in \mathcal{A}$. Define $\Delta_d^N = \Delta_d \cap \frac{1}{N}\mathbb{Z}^d$. In other words, $\underline{m} \in \Delta_d^N$ iff there exists a compatible $\underline{\alpha} \in \mathcal{A}^N$. In the above notation:

$$F(\underline{M}^N) |\underline{\alpha}\rangle = F(\underline{m}(\underline{\alpha})) |\underline{\alpha}\rangle. \quad (1.3)$$

A2. The transverse field B is stochastic: For any $\alpha, \beta \in \mathcal{A}$,

$$\lambda_{\alpha\beta} = \lambda_{\beta\alpha} \triangleq \langle\alpha|B|\beta\rangle \geq 0. \quad (1.4)$$

Furthermore, λ is an irreducible kernel on \mathcal{A} . Without loss of generality we shall assume that $\lambda \equiv 0$ on the diagonal.

1.3. An Example: Spin-s Models. The relation between the dimension d of \mathbb{X} and the half-integer spin \mathbf{s} is $d = 2\mathbf{s} + 1$. The set of classical labels is

$$\mathcal{A} = \{-\mathbf{s}, -\mathbf{s} + 1, \dots, \mathbf{s}\}.$$

The stochastic operators are $B_i = \lambda \mathbf{S}_i^x$. $\lambda \geq 0$ is the strength of the transverse field. Altogether, the Hamiltonian is

$$-\mathcal{H}_N = NF(M_{-\mathbf{s}}^N, M_{-\mathbf{s}+1}^N, \dots, M_{\mathbf{s}}^N) + \lambda \sum_i \mathbf{S}_i^x. \quad (1.5)$$

For instance, the case of p -body ferromagnetic interaction corresponds to

$$F(M_{-\mathbf{s}}^N, M_{-\mathbf{s}-1}^N, \dots, M_{\mathbf{s}}^N) = \left(\sum_{\alpha} \alpha M_{\alpha}^N \right)^p = \left(\frac{1}{N} \sum_i S_i^z \right)^p. \quad (1.6)$$

The operators S^x act (under convention that $|\mathbf{s}+1\rangle = |-\mathbf{s}-1\rangle = 0$) on \mathbb{X} as

$$S^x|\alpha\rangle = \frac{1}{2}\sqrt{s(s+1)-\alpha(\alpha-1)}|\alpha-1\rangle + \frac{1}{2}\sqrt{s(s+1)-\alpha(\alpha+1)}|\alpha+1\rangle \quad (1.7)$$

Consequently, the jump rates $\lambda_{\alpha\beta}$ are given by

$$\lambda_{\alpha\beta} = \begin{cases} \frac{\lambda}{2}\sqrt{s(s+1)-\alpha\beta}, & \text{if } |\alpha-\beta|=1 \\ 0, & \text{otherwise.} \end{cases} \quad (1.8)$$

1.4. The Result. In order to develop an asymptotic description of finite volume ground states we need to introduce some additional notation: For $\underline{m} \in \Delta_d^N$ set

$$c_N(\underline{m}) = \binom{N}{N\underline{m}} = \frac{N!}{\prod (Nm_{\alpha})!}.$$

The vectors $|\underline{m}\rangle \in \mathbb{X}_N$,

$$|\underline{m}\rangle \triangleq \frac{1}{\sqrt{c_N(\underline{m})}} \sum_{\underline{\alpha} \sim \underline{m}} |\underline{\alpha}\rangle$$

are normalized and orthogonal for different $\underline{m} \in \Delta_d^N$.

By Perron-Frobenius theorem and Lemma 2.1 below the ground state of \mathcal{H}_N is fully symmetrized, that is of the form

$$|h_N\rangle = \sum_{\underline{m} \in \Delta_d^N} h_N(\underline{m}) |\underline{m}\rangle, \quad (1.9)$$

and $h_N(\underline{m}) > 0$ for every $\underline{m} \in \Delta_d^N$ (see Subsection 2.1). Let us represent

$$h_N(\underline{m}) = e^{-N\psi_N(\underline{m})}. \quad (1.10)$$

It would be convenient to identify ψ_N with its linear interpolation (which is an element of the space of continuous functions $\mathcal{C}(\Delta_d)$).

Next introduce:

$$\begin{aligned} \mathcal{H}_0(\underline{m}, \underline{\theta}) &= \sum_{\alpha\beta} \sqrt{m_{\alpha}m_{\beta}} \lambda_{\alpha\beta} (\cosh(\theta_{\beta} - \theta_{\alpha}) - 1) \\ \mathcal{L}_0(\underline{m}, \underline{v}) &= \sup_{\underline{\theta}} \{(\underline{v}, \underline{\theta}) - \mathcal{H}_0(\underline{m}, \underline{\theta})\} \end{aligned} \quad (1.11)$$

For $\underline{m} \in \Delta_d$ define

$$V(\underline{m}) = \frac{1}{2} \sum_{\alpha, \beta} \lambda_{\alpha\beta} (\sqrt{m_{\beta}} - \sqrt{m_{\alpha}})^2 - F(\underline{m}). \quad (1.12)$$

Finally set $\lambda_\alpha = \sum_\beta \lambda_{\alpha\beta}$,

$$\begin{aligned} \mathcal{H}(\underline{m}, \underline{\theta}) &= \mathcal{H}_0(\underline{m}, \underline{\theta}) - V(\underline{m}) \\ &= \sum_{\alpha\beta} \sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta} \cosh(\theta_\beta - \theta_\alpha) - \sum_\alpha m_\alpha \lambda_\alpha + F(\underline{m}), \end{aligned} \quad (1.13)$$

and

$$\mathcal{L}(\underline{m}, \underline{v}) = \mathcal{L}_0(\underline{m}, \underline{v}) + V(\underline{m}).$$

In (1.13) above (\cdot, \cdot) is the usual scalar product on \mathbb{R}^d .

Theorem A. *Let E_N^1 be the bottom eigenvalue of \mathcal{H}_N . Set $\lambda = \sum_\alpha \lambda_\alpha$. Then the limit*

$$-\lambda + \mathbf{r}_1 \stackrel{\Delta}{=} \lim_{N \rightarrow \infty} \frac{E_N^1}{N} \quad (1.14)$$

exists. Moreover,

$$\mathbf{r}_1 = \min_{\underline{m}} V(\underline{m}). \quad (1.15)$$

Furthermore, the sequence $\{\psi_N\}$ is precompact in $\mathcal{C}(\Delta_d)$. Any subsequential limit ψ satisfies: For any $T \geq 0$ and any $\underline{m} \in \Delta_d$,

$$\psi(\underline{m}) = \inf_{\gamma : \gamma(T) = \underline{m}} \left\{ \psi(\gamma(0)) + \int_0^T \mathcal{L}(\gamma(t), \gamma'(t)) dt - T \mathbf{r}_1 \right\}, \quad (1.16)$$

where the infimum above is over all absolutely continuous curves $\gamma : [0, T] \rightarrow \Delta_d$. Moreover, the set \mathcal{M}_ψ of all local minima of ψ is a subset of $\mathcal{M}_V \stackrel{\Delta}{=} \operatorname{argmin}(V) \subset \operatorname{int}(\Delta_d)$.

Remark 1. *Hamiltonians $\mathcal{H}_0, \mathcal{H}$ are invariant under the shifts $\underline{\theta} \mapsto \underline{\theta} + c\mathbf{1}$, and, as a result, the Lagrangians $\mathcal{L}_0, \mathcal{L}$ are infinite whenever $(\underline{v}, \mathbf{1}) \neq 0$. Also, $\mathcal{L}_0(\underline{m}, 0) = 0 = \min \mathcal{L}_0(\underline{m}, \underline{v})$. Consequently, $\mathcal{L}(\underline{m}, 0) = V(\underline{m}) = -\mathcal{H}(\underline{m}, 0)$, and (1.15) could be rewritten as*

$$\mathbf{r}_1 = \min_{\underline{m}, \underline{v}} \mathcal{L}(\underline{m}, \underline{v}) = -\max_{\underline{m}} \mathcal{H}(\underline{m}, 0). \quad (1.17)$$

Remark 2. *Either of (1.15) and (1.16) unambiguously characterizes \mathbf{r}_1 , but not ψ . As we shall explain in the sequel, if ψ satisfies (1.16), then the weak KAM theory of Fathi [16] implies that ψ is a viscosity solution (see Subsection 2.7 for the precise statement) on $\operatorname{int}(\Delta_d)$ of the Hamilton-Jacobi equation*

$$\mathcal{H}(\underline{m}, \nabla \psi(\underline{m})) = -\mathbf{r}_1. \quad (1.18)$$

Note that since ψ is a function on Δ_d , the gradients $\nabla \psi$ lie in the subspace

$$\mathbb{R}_0^d = \{\underline{v} : (\underline{v}, \mathbf{1}) = 0\}. \quad (1.19)$$

In general there might be many viscosity solutions of (1.18) which comply with the conclusions of Theorem A. The solutions which are subsequential limits of $\{\psi_N\}$ are called *admissible*. Although we expect uniqueness of global admissible solutions for a large class of models, our approach does not offer a procedure for selecting the latter. The viscosity setup is important - at least for a large class of symmetric potentials the global admissible solutions are not smooth and develop shocks. A

proper selection procedure should be related to a more refined analysis of the low lying spectra of \mathcal{H}_N . As it was mentioned in the Introduction sharp asymptotics of eigenvalues and eigenfunctions in vicinity of potential wells were derived in a much more general context in [23–25]. In particular, it is explained therein how such asymptotics are related to (smooth) local solutions of (1.18). Implications of these results for a characterization of *global* admissible solutions is beyond the scope of this work and hopefully shall be addressed in full generality elsewhere. In the concluding Section 4 we work out a particular case of spin- $\frac{1}{2}$ models..

2. STRUCTURE OF THE THEORY

2.1. Spectrum of \mathcal{H}_N . Let \mathbb{X}_N^s be the sub-space of \mathbb{X}_N which consist of those vectors $|b\rangle$ which do not vanish under symmetrization. Namely, $|b\rangle = \sum_{\underline{\alpha}} a_{\underline{\alpha}} |\underline{\alpha}\rangle \in \mathbb{X}_N^s \setminus 0$, if $\sum_{\pi} a_{\pi \underline{\alpha}} \neq 0$ for some $\underline{\alpha} \in \mathcal{A}^N$, where π is a permutation of $\{1, \dots, N\}$ with $\pi \underline{\alpha}(i) = \underline{\alpha}(\pi_i)$. The sub-space \mathbb{X}_N^s is invariant for the Hamiltonian \mathcal{H}_N . The ground state of \mathcal{H}_N always belongs to \mathbb{X}_N^s . For the rest of the paper we shall work with the restriction of \mathcal{H}_N to \mathbb{X}_N^s .

All eigenfunctions of \mathcal{H}_N (restricted to \mathbb{X}_N^s) have mean-field representatives:

Lemma 2.1. *If E_N is an eigenvalue of \mathcal{H}_N , then there exists a function h_N on Δ_d^N such that $|h_N\rangle \triangleq \sum_{\underline{m} \in \Delta_d^N} h_N(\underline{m}) |\underline{m}\rangle$ is a corresponding eigenfunction:*

$$\mathcal{H}_N |h_N\rangle = E_N |h_N\rangle. \quad (2.1)$$

Proof. Let E_N be an eigenvalue of \mathcal{H}_N . Let $|b_N\rangle = \sum_{\underline{\alpha} \in \mathcal{A}^N} a_{\underline{\alpha}} |\underline{\alpha}\rangle \in \mathbb{X}_N^s$ be an eigenfunction corresponding to the eigenvalue E_N . Let $C(\underline{\alpha}, \underline{\beta}) = \langle \underline{\beta} | \hat{B} | \underline{\alpha} \rangle$ be the matrix elements of $\hat{B} \triangleq \sum_i B_i$. Thus, $\hat{B} |\underline{\alpha}\rangle = \sum_{\underline{\beta}} C(\underline{\alpha}, \underline{\beta}) |\underline{\beta}\rangle$. The eigenfunction equation is recorded as: $\forall \underline{\beta}$

$$\sum_{\underline{\alpha}} a_{\underline{\alpha}} C(\underline{\alpha}, \underline{\beta}) = (-E_N - F(\underline{m})) a_{\underline{\beta}}.$$

Note that $C(\underline{\alpha}, \underline{\beta}) = C(\pi \underline{\alpha}, \pi \underline{\beta})$. Consequently, since in addition $\underline{m}(\underline{\beta}) = \underline{m}(\pi \underline{\beta})$,

$$\sum_{\underline{\alpha}} a_{\pi \underline{\alpha}} C(\underline{\alpha}, \underline{\beta}) = (-E_N - F(\underline{m})) a_{\pi \underline{\beta}}.$$

Therefore, $|\pi b_N\rangle \triangleq \sum_{\underline{\alpha}} a_{\pi \underline{\alpha}} |\underline{\alpha}\rangle$ is also an eigenfunction. Since the sum $\sum_{\pi} a_{\pi \underline{\alpha}}$ does not change if we permute the entries of $\underline{\alpha}$, and since, by assumption $|b_N\rangle \in \mathbb{X}_N^0$, the claim follows with $|h_N\rangle = \sum_{\pi} |\pi b_N\rangle$. \square

2.2. Stochastic Representation. Let $\alpha(t)$ be the continuous time Markov chain on \mathcal{A} with jump rates $\lambda_{\alpha\beta}$. $\mathbb{P}_{\underline{\alpha}}^N$ is the path measure for N independent copies of such chain starting from $\underline{\alpha}$. Then the following representation of the entries of the density matrix holds [3, 22]: For any $T \geq 0$ and any $\underline{\alpha}, \underline{\beta} \in \mathcal{A}^N$

$$e^{-N\lambda T} \langle \underline{\beta} | e^{-T\mathcal{H}_N} | \underline{\alpha} \rangle = \mathbb{E}_{\underline{\alpha}}^N \exp \left\{ N \int_0^T F(\underline{m}(t)) dt \right\} \mathbb{I}_{\{\underline{\alpha}(T) = \underline{\beta}\}}. \quad (2.2)$$

Above $\underline{m}(t) = \underline{m}(\underline{\alpha}(t))$.

2.3. Mean Field Lumping. The process $\underline{m}_N(t) = \underline{m}(t) = \underline{m}(\underline{\alpha}(t))$ is a continuous time Markov chain on Δ_d^N with the generator

$$\mathcal{G}_N f(\underline{m}) = N \sum_{\alpha, \beta} m_\alpha \lambda_{\alpha\beta} \left(f \left(\underline{m} + \frac{\delta_\beta - \delta_\alpha}{N} \right) - f(\underline{m}) \right). \quad (2.3)$$

It is reversible with respect to the measure

$$\mu_N(\underline{m}) \triangleq \frac{c_N(\underline{m})}{d^N}. \quad (2.4)$$

Summing up in (2.2),

$$e^{-N\lambda T} \langle \underline{m}' | e^{-T\mathcal{H}_N} | \underline{m} \rangle = \sqrt{\frac{\mu_N(\underline{m})}{\mu_N(\underline{m}')}} \mathbb{E}_{\underline{m}}^N \exp \left\{ N \int_0^T F(\underline{m}(t)) dt \right\} \mathbb{1}_{\{\underline{m}(T) = \underline{m}'\}}, \quad (2.5)$$

for every $T \geq 0$ and every $\underline{m}, \underline{m}' \in \Delta_d^N$.

Using Girsanov's formula one can rewrite (2.5) in a variety of ways for different modifications of the jump rates in (2.3). Namely, let g be a positive function on Δ_d^N . Consider the modified rates

$$\lambda_{\alpha\beta}^{N,g} = \lambda_{\alpha\beta}^{N,g}(\underline{m}) = \frac{1}{g(\underline{m})} N m_\alpha \lambda_{\alpha\beta} g(\underline{m} + (\delta_\beta - \delta_\alpha)/N).$$

Let $\mathbb{P}_{\underline{m}}^{N,g}$ be the corresponding path measure. Then, the right hand side of (2.5) reads as

$$\sqrt{\frac{\mu_N(\underline{m})g(\underline{m})^2}{\mu_N(\underline{m}')g(\underline{m}')^2}} \mathbb{E}_{\underline{m}}^{N,g} \exp \left\{ N \int_0^T F_g(\underline{m}(t)) dt \right\} \mathbb{1}_{\{\underline{m}(T) = \underline{m}'\}}, \quad (2.6)$$

where

$$F_g(\underline{m}) = \sum_{\alpha\beta} \lambda_{\alpha\beta} m_\alpha \left(\frac{g(\underline{m} + (\delta_\beta - \delta_\alpha)/N)}{g(\underline{m})} - 1 \right) + F(\underline{m}). \quad (2.7)$$

A self-suggesting choice is

$$g(\underline{m}) = \frac{1}{\sqrt{\mu_N(\underline{m})}} \Rightarrow \lambda_{\alpha\beta}^{N,g} = N \sqrt{m_\alpha(m_\beta + 1/N)} \lambda_{\alpha\beta}. \quad (2.8)$$

For the rest of the paper we fix g as in (2.8). The corresponding generator

$$\mathcal{G}_N^g f(\underline{m}) = \sum_{\alpha, \beta} \lambda_{\alpha\beta}^{N,g} \left(f \left(\underline{m} + \frac{\delta_\beta - \delta_\alpha}{N} \right) - f(\underline{m}) \right). \quad (2.9)$$

is reversible with respect to the uniform measure on Δ_d^N . The function F_g in (2.7) equals to

$$F_g(\underline{m}) = -V(\underline{m}) + \Xi_N(\underline{m}), \quad (2.10)$$

where V is precisely the function defined in (1.12), and the correction

$$\Xi_N(\underline{m}) = \sum_{\alpha\beta} \lambda_{\alpha\beta} \sqrt{m_\alpha} \left(\sqrt{m_\beta + 1/N} - \sqrt{m_\beta} \right). \quad (2.11)$$

All together, (2.5) reads as

$$e^{-N\lambda T} \langle \underline{m}' | e^{-T\mathcal{H}_N} | \underline{m} \rangle = \mathbb{E}_{\underline{m}}^{N,g} \exp \left\{ -N \int_0^T (V - \Xi_N)(\underline{m}(t)) dt \right\} \mathbb{1}_{\{\underline{m}(T) = \underline{m}'\}}, \quad (2.12)$$

As we shall see below it happens to be convenient to work simultaneously with both representations (2.5) and (2.12).

Note that an immediate consequence of (2.12) is:

Lemma 2.2. *E_N is an eigenvalue of \mathcal{H}_N with $|u_N\rangle = \sum u_N(\underline{m})|\underline{m}\rangle$ being the corresponding normalized eigenfunction if and only if u_N is also an eigenfunction of $\mathcal{S}_N \triangleq \mathcal{G}_N^g + NF_g = \mathcal{G}_N^g - N(V - \Xi_N)$ with*

$$-R_N \triangleq -(N\lambda + E_N) \quad (2.13)$$

being the corresponding eigenvalue.

2.4. The Eigenfunction Equation. Assumption **A.2** and Perron-Frobenius theorem imply that \mathcal{H}_N has a non-degenerate ground state $|h_N\rangle = \sum_{\underline{m}} h_N(\underline{m})|\underline{m}\rangle$ with strictly positive entries $h_N(\underline{m}) > 0$. By Lemma 2.2, $h_N(\underline{m})$ is the principal eigenfunction of $\mathcal{G}_N^g + NF_g(\underline{m})$ with the corresponding top eigenvalue $-R_N^1 = -(N\lambda + E_N^1)$. The corresponding eigenfunction equation is: For any $T > 0$,

$$h_N(\underline{m}) = \mathbb{E}_{\underline{m}}^{N,g} \exp \left\{ -N \int_0^T (V - \Xi_N)(\underline{m}(t)) dt + TR_N^1 \right\} h_N(\underline{m}(T)), \quad (2.14)$$

By reversibility,

Lemma 2.3. *Functions $\{h_N\}$ satisfy: For every $T \geq 0$, N and $\underline{m} \in \Delta_d^N$*

$$h_N(\underline{m}) = \sum_{\underline{m}'} h_N(\underline{m}') \mathbb{E}_{\underline{m}'}^{N,g} \exp \left\{ -N \int_0^T (V - \Xi_N)(\underline{m}(t)) dt + TR_N^1 \right\} \mathbb{1}_{\{\underline{m}(T) = \underline{m}\}}. \quad (2.15)$$

2.5. Compactness and Large Deviations. For $\underline{m}, \underline{m}' \in \Delta_d^N$ define

$$Z_T^{N,g}(\underline{m}', \underline{m}) \triangleq \frac{1}{N} \log \mathbb{E}_{\underline{m}'}^{N,g} \exp \left\{ -N \int_0^T (V - \Xi_N)(\underline{m}(t)) dt \right\} \mathbb{1}_{\{\underline{m}(T) = \underline{m}\}}, \quad (2.16)$$

In the sequel we shall identify $Z_T^{N,g}(\cdot, \cdot)$ with its continuous interpolation on $\Delta_d \times \Delta_d$.

Let \mathcal{AC}_T be the family of all absolutely continuous trajectories $\gamma : [0, T] \mapsto \Delta_d$. For $\underline{m}, \underline{m}' \in \Delta_d$ define

$$Z_T^g(\underline{m}', \underline{m}) \triangleq - \inf_{\substack{\gamma(0) = \underline{m}', \gamma(T) = \underline{m} \\ \gamma \in \mathcal{AC}_T}} \int_0^T \mathcal{L}(\gamma(t), \gamma'(t)) dt, \quad (2.17)$$

where the Lagrangian \mathcal{L} was defined in (1.13).

Theorem 2.1. *For all T sufficiently large the sequence of functions $\{Z_T^{N,g}\}$ is equi-continuous on $\Delta_d \times \Delta_d$ and uniformly locally Lipschitz on $\text{int}(\Delta_d \times \Delta_d)$. Furthermore, for all T sufficiently large,*

$$\lim_{N \rightarrow \infty} \left| Z_T^{N,g}(\underline{m}', \underline{m}) - Z_T^g(\underline{m}', \underline{m}) \right| = 0 \quad (2.18)$$

simultaneously for all $\underline{m}, \underline{m}' \in \Delta_d$.

Note that the equi-continuity of $\{Z_T^{N,g}\}$ in Theorem 2.1 implies that the convergence in (2.18) is actually uniform. Consequently, $Z_T^g(\cdot, \cdot)$ is continuous on $\Delta_d \times \Delta_d$ and locally Lipschitz on $\text{int}(\Delta_d \times \Delta_d)$.

Theorem 2.1 is a somewhat standard statement. Its proof will be sketched in Subsection 3.1.

2.6. Lax-Oleinik Semigroup and Weak KAM. Recall the representation of the leading eigenfunction $h_N(\underline{m}) = e^{-N\psi_N(\underline{m})}$. In the sequel we shall identify ψ_N with its (continuous) interpolation on Δ_d ; $\psi_N \in \mathcal{C}(\Delta_d)$.

Theorem 2.2. *The sequence of numbers R_N^1/N is bounded in \mathbb{R} . The sequence of functions $\{\psi_N\}$ is equi-continuous on Δ_d and uniformly locally Lipschitz on $\text{int}(\Delta_d)$.*

Proof. Since R_N^1 is the Perron-Frobenius eigenvalue,

$$\frac{R_N^1}{N} = -\frac{1}{N} \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_{\underline{m}'}^{N,g} \exp \left\{ N \int_0^T F_g(\underline{m}(s)) ds \right\},$$

which is bounded since F_g is bounded on Δ_d . On the other hand, by (2.15), the equi-continuity and the uniform local Lipschitz property of $\{\psi_N\}$ is inherited from the corresponding properties of $\{Z_T^{N,g}\}$. \square

Proof of (1.14) and (1.16) of Theorem A: Theorems 2.1, 2.2 and Lemma 2.3 imply that any accumulation point $(\mathbf{r}, \psi) \in \mathbb{R} \times \mathcal{C}(\Delta_d)$ of the sequence $\{\frac{1}{N}R_N^1, \psi_N\}$ satisfies: ψ is locally Lipschitz on $\text{int}(\Delta_d)$ and

$$\psi(\underline{m}) = \inf_{\gamma(T)=\underline{m}} \left\{ \psi(\gamma(0)) + \int_0^T \mathcal{L}(\gamma(t), \gamma'(t)) dt \right\} - T\mathbf{r} \triangleq \mathcal{U}_T \psi(\underline{m}) - T\mathbf{r}. \quad (2.19)$$

for every $T \geq 0$ and each $\underline{m} \in \Delta_d$. Accumulation points of ψ_N are called *admissible solutions* of (2.19). Since \mathcal{U}_T is non-expanding on $\mathcal{C}(\Delta_d)$, validity of equation (2.19) unambiguously determines \mathbf{r} , which implies that the limit $\mathbf{r}_1 \triangleq \lim_{N \rightarrow \infty} \frac{R_N^1}{N}$ indeed exists. In view of (2.13) we, therefore, have established (1.14) and (1.16) of Theorem A. \square

2.7. Viscosity Solutions. Recall the definition (1.13) of $\mathcal{H}(\underline{m}, \underline{\theta})$. For \mathbf{r}_1 defined as above consider the Hamilton-Jacobi equation

$$\mathcal{H}(\underline{m}, \nabla \psi(\underline{m})) = -\mathbf{r}_1. \quad (2.20)$$

Definition 1. For $\underline{m} \in \Delta_d$ and $\psi \in \mathcal{C}(\Delta_d)$ define lower and upper sub-differentials

$$D_- \psi(\underline{m}) = \left\{ \xi \in \mathbb{R}_0^n : \liminf_{\underline{m}' \rightarrow \underline{m}} \frac{\psi(\underline{m}') - \psi(\underline{m}) - (\xi, \underline{m}' - \underline{m})}{|\underline{m}' - \underline{m}|} \geq 0 \right\}, \quad (2.21)$$

and, similarly, $D_+ \psi(\underline{m})$ with \liminf changed to \limsup and the sign of the inequality flipped.

A locally Lipschitz function ψ is said to be a viscosity supersolution of (2.20) at \underline{m} if $\mathcal{H}(\underline{m}, \xi) \geq -\mathbf{r}_1$ for every $\xi \in D_- \psi(\underline{m})$. Similarly, it is said to be a viscosity subsolution of (2.20) at \underline{m} if $\mathcal{H}(\underline{m}, \xi) \leq -\mathbf{r}_1$ for every $\xi \in D_+ \psi(\underline{m})$. ψ is viscosity solution of (2.20) at \underline{m} if it is both sub and super viscosity solution.

Theorem 2.3. If ψ is a weak-KAM solution (of (2.19) with $\mathbf{r} = \mathbf{r}_1$), then it is a viscosity solution of (2.20) on $\text{int}(\Delta_d)$.

The proof of Theorem 2.3 is relegated to Subsection 3.2.

2.8. Minima of ψ .

Theorem 2.4. Let ψ be a weak KAM solution of (2.19). Then all local minima of ψ lie in the interior $\text{int}(\Delta_d)$.

Theorem 2.4 will be proved in Subsection 3.3

2.9. Stochastic Representation of the Ground State. The eigenfunction equation (2.14) defines a Markovian semi-group

$$\widehat{\mathbb{E}}_T^N f(\underline{m}) = \frac{1}{h_N(\underline{m})} \mathbb{E}_{\underline{m}}^{N,g} \exp \left\{ N \int_0^T F_g(\underline{m}(t)) dt + TR_N^1 \right\} h_N(\underline{m}(T)) f(\underline{m}(T)). \quad (2.22)$$

This corresponds to continuous time Markov chain with the generator

$$\widehat{\mathcal{G}}_N^g f(\underline{m}) = \frac{1}{h_N(\underline{m})} \sum_{\alpha, \beta} \lambda_{\alpha\beta}^{N,g} h_N \left(\underline{m} + \frac{\delta_\beta - \delta_\alpha}{N} \right) \left(f \left(\underline{m} + \frac{\delta_\beta - \delta_\alpha}{N} \right) - f(\underline{m}) \right). \quad (2.23)$$

In the sequel we shall refer to $\widehat{\mathcal{G}}_N^g$ as to the generator of the ground state chain.

Lemma 2.4. The generator $\widehat{\mathcal{G}}_N^g$ is reversible with respect to the probability measure $\nu_N(\underline{m}) \triangleq h_N^2(\underline{m}) = e^{-2N\psi_N(\underline{m})}$. Furthermore, E_N is an eigenvalue of \mathcal{H}_N if and only if $E_N^1 - E_N$ is an eigenvalue of $\widehat{\mathcal{G}}_N^g$.

It is straightforward to check that $\widehat{\mathcal{G}}_N^g$ satisfies the detailed balance condition with respect to ν_N . It is equally straightforward to see from (2.22) that g_N is an eigenfunction of $\mathcal{G}_N^g + NF_g$, and hence by Lemma 2.2 of \mathcal{H}_N , if and only if g_N/h_N is an eigenfunction of $\widehat{\mathcal{G}}_N^g$.

3. PROOFS

3.1. Proof of Theorem 2.1. Consider the family of processes $\{\underline{m}(\cdot) = \underline{m}_N(\cdot)\}$ with generator \mathcal{G}_N^g defined in (2.9). We shall identify \underline{m}_N with its linear interpolation. For each $T > 0$, the family $\{\underline{m}_N(\cdot)\}$ is exponentially tight on $\mathbf{C}_{0,T}(\Delta_d)$.

Recall the definition of \mathcal{L}_0 in (1.11). One can follow the approach of [15] and to combine the Large Deviation Principle for projective limits [13] with the inverse contraction principle of [21] in order to conclude:

Lemma 3.1. *For each $T > 0$ and every initial condition $\underline{m} \in \Delta_d$ (where for each N we identify \underline{m} with its discretization $\lfloor N\underline{m} \rfloor / N \in \Delta_d^N$) the family of processes $\{\underline{m}(t)\}$ satisfy a large deviations principle on $\mathbf{C}_{0,T}(\Delta_d)$ with the following good rate function*

$$I_T(\gamma) = \begin{cases} \int_0^T \mathcal{L}_0(\gamma(s), \gamma'(s)) ds, & \text{if } \gamma \text{ is absolutely continuous} \\ & \text{and } \gamma(0) = \underline{m}. \\ \infty, & \text{otherwise.} \end{cases} \quad (3.1)$$

By the upper bound in Varadhan's lemma,

$$\limsup_{N \rightarrow \infty} Z_T^{N,g}(\underline{m}', \underline{m}) \leq Z_T^g(\underline{m}', \underline{m})$$

On the other hand, by the lower bound in Varadhan's lemma, for each $\delta > 0$

$$Z_T^g(\underline{m}', \underline{m}) \leq \liminf_{N \rightarrow \infty} \sup_{|\underline{m}_1 - \underline{m}| < \delta} Z_T^{N,g}(\underline{m}', \underline{m}_1).$$

Therefore, (2.18) is a consequence of the claimed continuity of $\underline{m} \rightarrow Z_T^{N,g}(\underline{m}', \underline{m})$. Let us proceed with establishing the asserted continuity properties of the family $Z_T^{N,g}(\cdot, \cdot)$. By reversibility,

$$Z_T^{N,g}(\underline{m}, \underline{m}') = Z_T^{N,g}(\underline{m}', \underline{m}), \quad (3.2)$$

so it would be enough to explore those in the second variable only.

An equivalent task is to check continuity properties of

$$Z_T^N(\underline{m}', \underline{m}) \triangleq \frac{1}{N} \log \mathbb{E}_{\underline{m}'}^N \exp \left\{ N \int_0^T F(\underline{m}(t)) dt \right\} \mathbb{1}_{\{\underline{m}(T) = \underline{m}\}}, \quad (3.3)$$

Indeed, by (2.6)

$$Z_T^{N,g}(\underline{m}, \underline{m}') = \sqrt{\frac{\mu_N(\underline{m})}{\mu_N(\underline{m}')}} Z_T^N(\underline{m}, \underline{m}')$$

Now, under \mathbb{P}_N the process $\underline{m}_N(t)$ is a super-position of N independent particles which hop on the finite set \mathcal{A} with irreducible rates $\lambda_{\alpha\beta}$. Since F is bounded, the following claim is straightforward:

Lemma 3.2. *There exist $T_0 > 0$, $\epsilon_0 > 0$ and a constant $c_1 < \infty$ such that*

$$e^{NZ_{T-\epsilon}^N(\underline{m}', \underline{m})} \geq e^{-c_1 N \epsilon} e^{NZ_T^N(\underline{m}', \underline{m})}. \quad (3.4)$$

uniformly in N , $T \geq T_0$, $\epsilon \leq \epsilon_0$, $\underline{m}', \underline{m} \in \Delta_d$ and N .

Indeed, trajectories $\underline{m}(\cdot)$ on $[0, T]$ and trajectories $\tilde{m}(\cdot)$ on $[0, T - \varepsilon]$: are related by the following one to one map: $\tilde{m}(t) = \underline{m}(t \frac{T}{T-\varepsilon})$. Since for some $c_2 = c_2(T_0) > 0$, up to exponentially small factors, the total number of jumps of all the particles is at most $c_2 NT$, the Radon - Nikodým derivative is under control and (3.4) follows. \square

As a result, for any T, ϵ as above, and for any $\underline{m}', \underline{m}^1, \underline{m}^2 \in \Delta_d$,

$$Z_T^N(\underline{m}', \underline{m}^2) \geq Z_T^N(\underline{m}', \underline{m}^1) - c_1 \epsilon + Z_\epsilon^N(\underline{m}^1, \underline{m}^2). \quad (3.5)$$

Fix $\underline{m}^1, \underline{m}^2$ and define $\mathcal{A}_+ = \mathcal{A}_+(\underline{m}^1, \underline{m}^2) = \{\alpha : m_\alpha^1 > m_\alpha^2\}$. For $\alpha \in \mathcal{A}_+$ define $\delta_\alpha = m_\alpha^1 - m_\alpha^2$. One way to drive $\underline{m}(\cdot)$ from \underline{m}^1 to \underline{m}^2 during ϵ units of time is to choose $N\delta_\alpha$ particles out of Nm_α^1 for each $\alpha \in \mathcal{A}_+$, and to redistribute them during ϵ units of time into $\mathcal{A} \setminus \mathcal{A}_+$, without touching the rest of the particles. There is an obvious uniform lower bound $c_3 \epsilon^n$ that a particle starting at the state α will be at state β at time ϵ . We infer:

$$e^{NZ_\epsilon^N(\underline{m}^1, \underline{m}^2)} \geq e^{-(\max_\alpha \sum_\beta \lambda_{\alpha, \beta} - \min F)\epsilon N} \prod_{\alpha \in \mathcal{A}_+} \binom{Nm_\alpha^1}{N\delta_\alpha} (c_3 \epsilon^n)^{N\delta_\alpha}. \quad (3.6)$$

Hence,

$$Z_\epsilon^N(\underline{m}^1, \underline{m}^2) \geq -c_4 \epsilon - c_5 \sum_{\alpha \in \mathcal{A}_+} \delta_\alpha \left(d \log \frac{1}{\epsilon} - \log \frac{m_\alpha^1}{\delta_\alpha} \right). \quad (3.7)$$

Both, the equi-continuity of $\underline{m} \rightarrow Z_T^N(\underline{m}', \underline{m})$ on Δ_d and its uniform local Lipschitz property on $\text{int}(\Delta_d)$ readily follow from (3.5) and (3.7). \square

3.2. Proof of Theorem 2.3. We follow the approach of [16]: Let $\underline{m} \in \text{int}(\Delta)$ and assume that u is a smooth function such that $\{\underline{m}\} = \text{argmin}\{u - \psi\}$ in a neighbourhood of \underline{m} . Then,

$$u(\underline{m}) \leq u(\gamma(-t)) + \int_{-t}^0 \mathcal{L}(\gamma, \gamma') ds - r_1 t$$

for any $t \geq 0$ and for any smooth curve γ with $\gamma(0) = \underline{m}$. Let $\underline{v} = \gamma'(0)$. Then,

$$\nabla u(\underline{m}) \cdot \underline{v} - \mathcal{L}(\underline{m}, \underline{v}) \leq -r_1.$$

Since the above holds for any $\underline{m} \in \mathbb{R}_0^n$, $\mathcal{H}(\underline{m}, \nabla u(\underline{m})) \leq -r_1$ follows.

In order to check that ψ is a super-solution, note that by the upper and lower bounds on the Lagrangian \mathcal{L} derived in the Appendix, and by the local Lipschitz property of (bounded and continuous) ψ the minimum

$$\min_{\gamma(t_0)=\underline{m}} \left\{ \psi(\gamma(0)) + \int_0^{t_0} (\mathcal{L}(\gamma, \gamma') - r_1) ds \right\}$$

is attained at some γ_* with $\gamma_*(0) = \underline{m}'$ in a δ_0 -neighbourhood of \underline{m} , for all t_0 and δ_0 appropriately small. As it is explained in the Appendix, the minimizing curve γ_* is C^∞ and stays inside $\text{int}(\Delta_d)$. Evidently,

$$\psi(\underline{m}) = \psi(\gamma_*(t)) + \int_t^{t_0} (\mathcal{L}(\gamma_*, \gamma'_*) - r_1) ds$$

for every $t \in [0, t_0]$. Assume that u is smooth and $\operatorname{argmax} \{u - \psi\} = \{\underline{m}\}$ in a δ_0 neighbourhood of \underline{m} . Then,

$$u(\underline{m}) = u(\gamma_*(t_0)) \geq u(\gamma_*(t)) + \int_t^{t_0} (\mathcal{L}(\gamma_*, \gamma'_*) - r_1) ds$$

for every $t \in [0, t_0]$. Set $\underline{v} = \gamma'_*(t_0)$. We infer:

$$\nabla u(\underline{m}) \cdot \underline{v} \geq \mathcal{L}(\underline{m}, \underline{v}) - r_1.$$

Consequently, $\mathcal{H}(\underline{m}, \nabla u(\underline{m})) \geq -r_1$. \square

3.3. Proof of (1.15) of Theorem A and Theorem 2.4. By Theorem 2.2 and since $\nu_N(\underline{m}) = e^{-2N\psi_N(\underline{m})}$ it follows that

$$\min_{\underline{m}} \psi(\underline{m}) = 0.$$

Let us rewrite (1.16) as

$$\psi(\underline{m}) = \inf_{\gamma(T)=\underline{m}} \left\{ \psi(\gamma(0)) + \int_0^T (\mathcal{L}(\gamma, \gamma') - r_1) dt \right\} \quad (3.8)$$

Since $V(\underline{m}) = \mathcal{L}(\underline{m}, 0) \leq \mathcal{L}(\underline{m}, \underline{v})$, the above might be possible only if $r_1 = \min_{\underline{m}} V(\underline{m})$.

Furthermore, the Lagrangian \mathcal{L} is uniformly super-linear in the second variable: By (A.1) of the Appendix for every $C > 0$ and $\delta > 0$ we can find $T > 0$ such that

$$\inf_{\operatorname{diam}(\gamma) > \delta} \int_0^T (\mathcal{L}(\gamma, \gamma') - r_1) dt \geq C.$$

Which means that for $C > \max \psi$, the contribution to (3.8) for γ -s with the diameter larger than δ could be ignored.

Let, therefore, $\operatorname{diam}(\gamma) \leq \delta$. By (1.17),

$$\int_0^T (\mathcal{L}(\gamma, \gamma') - r_1) dt \geq T \min_{\underline{m}' \in \gamma} (V(\underline{m}') - r_1) \quad (3.9)$$

We infer:

$$\psi(\underline{m}) \geq \min_{|\underline{m}' - \underline{m}| \leq \delta} \psi(\underline{m}') + T \min_{|\underline{m}' - \underline{m}| \leq \delta} (V(\underline{m}') - r_1). \quad (3.10)$$

The claim of Theorem 2.4 follows as soon as we notice that all the minima of $\underline{m} \mapsto V(\underline{m})$ belong to $\operatorname{int}(\Delta_d)$. \square

4. RESULTS FOR SPIN- $\frac{1}{2}$ MODEL

For spin-s models (1.5)

$$r_1 = \min_{\underline{m}} V(\underline{m}) = \min_{\underline{m}} \left\{ \frac{\lambda}{4} \sum_{|\alpha - \beta| = 1} \sqrt{s(s+1) - \alpha\beta} (\sqrt{m_\alpha} - \sqrt{m_\beta})^2 - F(\underline{m}) \right\}. \quad (4.1)$$

In spin- $\frac{1}{2}$ case it is convenient to take $\{-1, 1\}$ instead of $\{-\frac{1}{2}, \frac{1}{2}\}$ as a set of classical labels for Spin- $\frac{1}{2}$ Model. The Hamiltonian is given by

$$-\mathcal{H}_N = NF(M_{-1}^N, M_1^N) + \lambda \sum_i \hat{\sigma}_i^x$$

where

$$\hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.2)$$

In this notation $\hat{\sigma}^z|\alpha\rangle = \alpha|\alpha\rangle$ and $\hat{\sigma}^x|\alpha\rangle = |-\alpha\rangle$ for $\alpha = \pm 1$.

The simplex Δ_d is just a segment $\{(m_{-1}, m_1) : m_{-1} + m_1 = 1\}$, parametrized by a single variable $m = m_1 - m_{-1} \in [-1, 1]$. Any vector $\underline{\theta} \in \mathbb{R}_0^2$ is of the form $\underline{\theta} = (\theta, -\theta)$. Define $F(m) = F(\frac{1-m}{2}, \frac{1+m}{2})$. Thus, in terms of m and θ , the Hamiltonian in (1.13) is

$$\mathcal{H}(m, \theta) = \lambda\sqrt{1-m^2} \cosh(2\theta) - \lambda + F(m). \quad (4.3)$$

Consequently, the effective potential

$$V(m) = -\mathcal{H}(m, 0) = \lambda - \left(\lambda\sqrt{1-m^2} + F(m) \right), \quad (4.4)$$

and the asymptotic leading eigenvalue r_1 is given by

$$r_1 = \min_{m \in (-1, 1)} V(m) = \lambda - \max_{m \in [-1, 1]} \left\{ \lambda\sqrt{1-m^2} + F(m) \right\}. \quad (4.5)$$

4.1. Minima of V . In order to explore the minimization problem (4.5) it would be convenient to represent $F(m) = -G(\text{sign}(m)\sqrt{1-m^2})$. Then,

$$r_1 = \lambda - \max_{-1 \leq t \leq 1} \{ \lambda|t| - G(t) \}. \quad (4.6)$$

Indeed, as it clearly seen from (4.5) (and as it follows in general by Theorem 2.4), all the minima of V belong to $(-1, 1)$, a possible jump discontinuity of G at zero (if $F(-1) \neq F(1)$) plays no role for the computation of maxima. Note also that $G(-1) = G(1) = F(0)$.

Let

$$\mathcal{M}_\lambda = \text{argmim}(V) = \{m \in (-1, 1) : \mathcal{H}(m, 0) = -r_1\}.$$

In other words, let $\mathcal{T}_\lambda \subset [-1, 1]$ be the set of maximizers in (4.6). We set $\mathcal{T}_\lambda = \mathcal{T}_\lambda^+ \cup \mathcal{T}_\lambda^-$, where $\mathcal{T}_\lambda^+ = \mathcal{T}_\lambda \cap (0, 1)$. Then,

$$m \in \mathcal{M}_\lambda \Leftrightarrow t = \text{sign}(m)\sqrt{1-m^2} \in \mathcal{T}_\lambda^{\text{sign}(m)}. \quad (4.7)$$

By assumption **A1**, G is a polynomial of finite degree on each of the intervals $[-1, 0]$ and $(0, 1]$, so the set \mathcal{T}_λ is finite, and its maximal cardinality is $\deg(G) - 1$. Various options are depicted on Figure 1 (for simplicity we depict only the $(0, 1]$ interval and, accordingly, the set \mathcal{T}_λ^+):

(a) First of all there exists $\lambda_c \in [0, \infty)$, such that $\mathcal{T}_\lambda = \{\pm 1\}$ on (λ_c, ∞) .

(b) If $\lambda_c > 0$, then \mathcal{T}_{λ_c} still contains ± 1 . It could happen, however, that $\mathcal{T}_{\lambda_c} = \{-1, t_1, \dots, t_k, 1\}$ contains at most $k \leq \deg(G)/2$ other points. In the latter case $(-1, 0) \cup (0, 1)$ necessarily contains at least $2k$ inflection points of G .

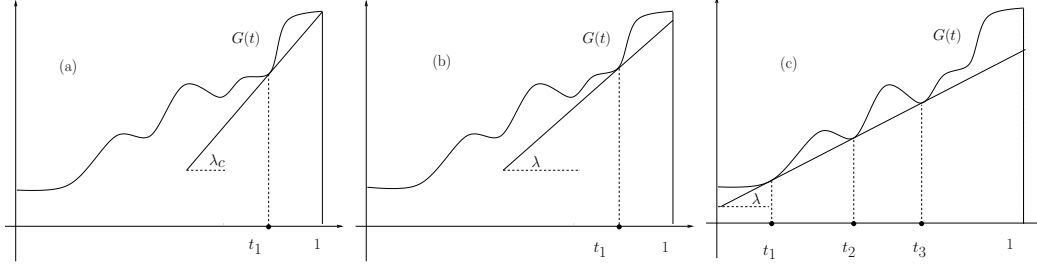


FIGURE 1. (a) The critical λ_c and $\mathcal{T}_{\lambda_c}^+ = \{t_1, 1\}$. (b) \mathcal{T}_{λ}^+ is a singleton. (c) $\mathcal{T}_{\lambda}^+ = \{t_1, t_2, t_3\}$. There are at least two inflection points of G on (t_1, t_3)

(c) There might be other exceptional values of $\lambda < \lambda_c$ for which either of $\mathcal{T}_{\lambda}^{\pm}$ is not a singleton. If, for instance $\mathcal{T}_{\lambda}^+ = \{t_1^{\lambda}, \dots, t_k^{\lambda}\}$ is not a singleton, then the interval $(t_1^{\lambda}, t_k^{\lambda})$ contains at least $2(k-1)$ inflection points of G . Since there are at most $\deg(G) - 2$ inflection points all together, and since intervals spanned by different $\mathcal{T}_{\lambda}^{\pm}$ are disjoint, we infer that $\mathcal{T}_{\lambda}^{\pm}$ is not a singleton for at most $\deg(G)/2$ values of λ .

Values of λ for which the cardinality of \mathcal{T}_{λ} changes correspond to first order phase transitions in the ground state.

4.2. Ferromagnetic p -body interaction. In the usual Curie-Weiss case with pair interactions $G(t) = \frac{1}{2}(t^2 - 1)$, so that $r_1 = -\frac{(\lambda-1)^2}{2}$ if $\lambda \leq 1$, and, accordingly, $r_1 = 0$ if $\lambda \geq 1$. For $\lambda \geq 1$ the set $\mathcal{M}_{\lambda} = \{0\}$. For $\lambda \in (0, 1)$, $\mathcal{M}_{\lambda} = \{\pm\sqrt{1-\lambda^2}\}$. No first order transition occurs.

In the $p > 2$ -body ferromagnetic interaction case (1.6) the function

$$G(t) = -\text{sign}(t)^p (1 - t^2)^{p/2}.$$

For odd p maximizers of $\lambda|t| - G(t)$ always lie in $(0, 1]$. For even p the set \mathcal{M}_{λ} is symmetric. Thus in either case it is enough to consider

$$r_1 = \lambda - \max_{t \in (0, 1]} \left\{ \lambda t + (1 - t^2)^{p/2} \right\}. \quad (4.8)$$

The crucial difference between the Curie-Weiss case $p = 2$ and $p > 2$ is that in the latter situation, $G'(1) = 0$, and G contains an inflection point $t_p = \sqrt{\frac{1}{p-1}}$ inside $(0, 1)$. An easy computation reveals that for $p > 2$,

$$\lambda_c = \frac{p}{p-1} \left(1 - \frac{1}{(p-1)^2} \right)^{\frac{p-1}{2}} \quad \text{and} \quad \mathcal{T}_{\lambda_c}^+ = \left\{ \frac{1}{p-1}, 1 \right\}. \quad (4.9)$$

Accordingly, for even p ,

$$\mathcal{M}_{\lambda_c} = \{0, \pm \hat{m}\} = \left\{ 0, \pm \sqrt{\frac{p(p-2)}{(p-1)^2}} \right\}, \quad (4.10)$$

whereas for odd p ; $\mathcal{M}_{\lambda_c} = \{0, \hat{m}\} = \left\{0, \sqrt{\frac{p(p-2)}{(p-1)^2}}\right\}$. This is precisely formula (14) of [5]. For $\lambda > \lambda_c$ the set $\mathcal{M}_\lambda = \{0\}$. For $\lambda < \lambda_c$ there exists $m^* = m^*(\lambda, p) \in \left(\sqrt{\frac{p(p-2)}{(p-1)^2}}, 1\right)$ such that the set \mathcal{M}_λ is a singleton $\{m^*\}$ in the odd case, whereas $\mathcal{M}_\lambda = \{\pm m^*\}$ in the even case. Thus, for mean-field models with p -body interaction, λ_c is the only value at which first order transition in the ground state occurs.

4.3. Asymptotic ground states. Let us return to general polynomial interactions F . Fix $m \in (-1, 1)$ and consider the equation,

$$\mathcal{H}(m, \theta) = -r_1. \quad (4.11)$$

The Hamiltonian $\mathcal{H}(m, \cdot)$ is strictly convex and symmetric. Hence, if $m \in \mathcal{M}_\lambda$, then $\theta = 0$ is the unique solution. If $m \notin \mathcal{M}_\lambda$, then necessarily $\mathcal{H}(m, 0) < -r_1$. Hence, there exist $\theta(m) > 0$, such that $\pm\theta(m)$ are the unique solutions to (4.11). If F is symmetric, then $\theta(-m) = -\theta(m)$. In any case, however, the following holds:

Let ψ be an admissible solution of $\mathcal{H}(m, \psi') = -r_1$. Since ψ is locally Lipschitz, it is a.e. differentiable. Consequently, $\psi'(m) = \pm\theta(m)$ a.e. on $(-1, 1)$. The proposition below relies only on the fact that ψ is a viscosity solution on $(-1, 1)$.

Proposition 4.1. *Record $\mathcal{M}_\lambda = \{-1 < m_1, \dots, m_k < 1\}$ in the increasing order. Set $m_0 = -1$ and $m_{k+1} = 1$. Then on each of the intervals $[m_\ell, m_{\ell+1}]$ the gradient ψ' is of the following form: There exists $m_\ell^* \in [m_\ell, m_{\ell+1}]$, such that:*

$$\psi' = \theta \text{ on } [m_\ell, m_\ell^*) \quad \text{and} \quad \psi' = -\theta \text{ on } (m_\ell^*, m_{\ell+1}]. \quad (4.12)$$

Proof. It would be enough to prove the following: If $m \in (m_\ell, m_{\ell+1})$ and $\psi'(m) = \theta(m)$, then for any $n \in (m_\ell, m)$,

$$\psi(n) = \psi(m) - \int_n^m \theta(t) dt. \quad (4.13)$$

Recall that since ψ is a viscosity solution on $(-1, 1)$, then

$$\liminf_{\varepsilon \rightarrow 0} \frac{\psi(n^* + \varepsilon) - \psi(n^*)}{|\varepsilon|} \geq \theta \Rightarrow \theta \notin (-\theta(n^*), \theta(n^*)). \quad (4.14)$$

for any $n^* \in (-1, 1)$. We shall show that if (4.13) is violated for some $n \in (m_\ell, m)$, then (4.14) is violated as well in the sense that there exists $n^* \in (n, m)$ and $\theta \in (-\theta(n^*), \theta(n^*))$ such that the right hand side of (4.14) holds.

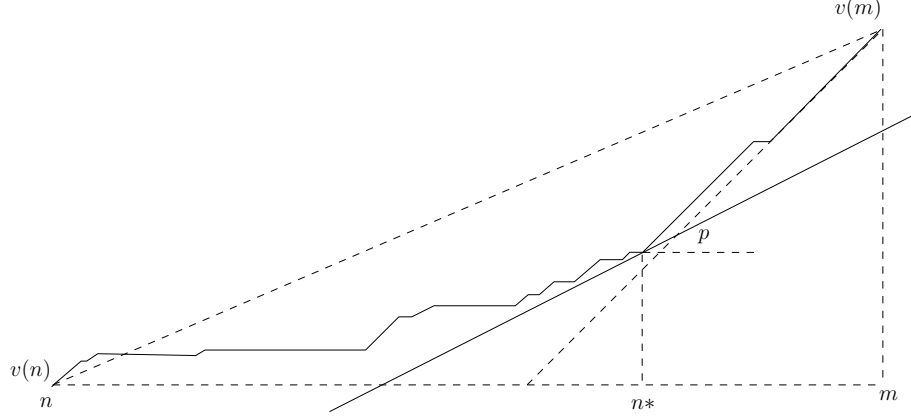
Indeed, since $\psi'(t) = \pm\theta(t)$ a.e. on $(-1, 1)$ it always holds that

$$\psi(n) \geq \psi(m) - \int_n^m \theta(t) dt$$

Let us assume strict inequality. For $k \in [n, m]$ define

$$v(k) = \int_n^k \frac{\psi'(t) + \theta(t)}{2\theta(t)} dt.$$

By construction $v(n) = 0$, $v(m) \triangleq p(m - n) < (m - n)$ and $v'(m) = 1$. There is no loss of generality to assume that $p > 0$. Hence there exists $n^* \in (n, m)$ such that

FIGURE 2. $p \in \partial v(n^*)$.

$p \in \partial v(n^*)$ (see Figure 2). By continuity of $\theta(m)$ this would mean that

$$\liminf_{\varepsilon \rightarrow 0} \frac{\psi(n^* + \varepsilon) - \psi(n^*)}{|\varepsilon|} \geq p\theta(n^*) - (1 - p)\theta(n^*) = (2p - 1)\theta(n^*).$$

Since for any inner point $n^* \in (m_\ell, m_{\ell+1})$; $\theta(n^*) > 0$, we arrived to a contradiction. \square

Remark 3. Note that Proposition 4.1 implies that ground states ψ with more than one local minimum necessarily develop shocks.

If $m_\ell < m_\ell^* < m_{\ell+1}$ from Proposition 4.1, then m_ℓ^* is a local maximum of ψ , and m_ℓ^* is a shock location for the stationary Hamilton-Jacobi equation $\mathcal{H}(m, \psi) = -r_1$.

If ψ is, in addition, a weak KAM solution (in particular, if ψ is admissible), then $\operatorname{argmin}\{\psi\} \subseteq \mathcal{M}_\lambda$. Consequently, by Theorem 2.4, in the latter case, $\psi' = -\theta$ on $(-1, m_1)$ and $\psi' = \theta$ on $(m_k, 1)$.

Admissible solutions are always normalized in the sense that $\min \psi = 0$ and the minimum is attained on $\mathcal{M}_\lambda \subset (-1, 1)$. It follows that admissible are uniquely defined in the following two cases:

CASE 1. The set $\mathcal{M}_\lambda = \{m^*\}$ is a singleton. Then, $\psi' = -\theta$ on $(-1, m^*)$ and $\psi' = \theta$ on $(m^*, 1)$. Consequently,

$$\psi(m) = \left| \int_{m^*}^m \theta(t) dt \right|. \quad (4.15)$$

CASE 2. The interaction F is symmetric and $\mathcal{M}_\lambda = \{\pm m^*\}$. Then, ψ is also symmetric; $\psi' = \theta$ on $(-m^*, 0) \cup (m^*, 1)$ and $\psi' = -\theta$ on $(-1, 0) \cup (0, m^*)$. That is $\psi(m) = \psi(-m)$ and ψ is still given by (4.15) for $m \geq 0$. Note that in this case ψ' has a jump at $m = 0$.

Let Λ^c be the set of λ which do not fall into one of the two cases above. As we have seen in Subsection 4.2, $\Lambda_c = \emptyset$ in the case of Curie-Weiss model, and $\Lambda_c = \{\lambda_c\}$ (see (4.9)) for general p -body interaction.

4.4. Multiple Wells. We shall refer to $\lambda \in \Lambda_c$ as to the case of multiple wells. Note first of all that there is a continuum of normalized solution of (1.16) as soon as the cardinality $|\mathcal{M}_\lambda| \geq 2$. Indeed, it is easy to see that any normalized ψ which complies with the conclusion of Proposition 4.1 will be a solution to (1.16).

One needs, therefore, an additional criterion to determine locations of shocks $\{m_\ell^*\}$ or, equivalently, to determine values $\{\psi(m_\ell)\}$ for admissible solutions. It would be tempting to derive location of shocks by some natural limiting procedure via stabilization of shock propagation along Rankine-Hugoniot curves. Since however, we arrived to (1.16) directly from the eigenvalue equation without recourse to a finite horizon problem, it was not clear to us which limit to consider. Our selection of admissible solutions to (1.16) is based on a refined asymptotic analysis of Dirichlet eigenvalues in a vicinity of points belonging to the set \mathcal{M}_λ . Namely, a point $m_\ell \in \mathcal{M}_\lambda$ can be local minima of an admissible solution ψ only if there is an exponential splitting of the corresponding bottom eigenvalues. Precise result is formulated in Proposition 4.2 below.

The results of [23, 24, 24]. enable to explore asymptotic expansions of such eigenvalues with any degree of precision. In the simplest case we deduce the following:

Corollary 4.1. *Assume that*

$$\min_{m \in \mathcal{M}_\lambda} \chi_0(m) \triangleq \min_{m \in \mathcal{M}_\lambda} \left\{ \frac{\lambda}{1 - m^2} - \sqrt{1 - m^2} F''(m) \right\} \quad (4.16)$$

is attained at either a unique point m^ (non-symmetric potentials) or at a unique couple $\pm m^*$ (symmetric potentials). Then there is a unique admissible solution ψ , which is still given by (4.15).*

For instance, in the critical ($\lambda = \lambda_c$) case of $p > 2$ body interaction, a substitution of (4.9) and (4.10) yields:

$$\chi_0(0) = \lambda_c \quad \text{and} \quad \chi_0(\hat{m}) = (p - 2)(p - 1)\lambda_c. \quad (4.17)$$

In other words, $\chi_0(0) < \chi_0(\hat{m})$, for any $p > 2$ and $\lambda = \lambda_c(p)$. Consequently, even at $\lambda = \lambda_c$ there is still a unique admissible solution $\psi(m) = |\int_0^m \theta(t) dt|$ with the unique minimum at $m^* = 0$.

We explain Corollary 4.1 in the concluding paragraph of this Section.

Spectral Asymptotics and the Set \mathcal{M}_λ . Assume that $\lambda \in \Lambda_c$ and, as before, denote $\mathcal{M} = \{m_1, \dots, m_k\}$.

Lemma 4.1. *For any $\delta > 0$ there exists $\epsilon > 0$ such that*

$$\min_{d(m, \mathcal{M}_\lambda) \geq \delta} \psi(m) \geq \epsilon, \quad (4.18)$$

uniformly in normalized admissible solutions of (1.16).

Proof. Let $m \in (m_l, m_{l+1})$. By Proposition 4.1

$$\psi(m) \geq \min \left\{ \psi(m_l) + \int_{m_l}^m \theta(t) dt, \psi(m_{l+1}) + \int_m^{m_{l+1}} \theta(t) dt \right\},$$

and (4.18) follows. \square

In the sequel $h_N = e^{-N\psi_N}$ is the Perron-Frobenius eigenfunction of $\mathcal{G}_N^g + NF_g \triangleq \mathcal{S}_N$; $\mathcal{S}_N h_N = -R_N^1 h_N$. Recall:

$$\begin{aligned} \mathcal{S}_N f(m) &= N(F(m) - \lambda)f(m) \\ &\quad + \frac{N\lambda}{2} \sqrt{(1-m)(1+m+\frac{2}{N})} f\left(m + \frac{2}{N}\right) \\ &\quad + \frac{N\lambda}{2} \sqrt{(1+m)(1-m+\frac{2}{N})} f\left(m - \frac{2}{N}\right) \end{aligned} \quad (4.19)$$

Pick $0 < \delta < \frac{1}{4} \min_l |m_{l+1} - m_l|$. Let $1 \equiv \sum_0^k \chi_l$ be a partition of unity satisfying: For $l = 1, \dots, k$

$$\chi_l \equiv 1 \text{ on } I_\delta(m_l) \quad \text{and} \quad \chi_l \equiv 0 \text{ on } I_{2\delta}^c(m_l).$$

Above $I_\eta(m)$ is the interval $[m - \eta, m + \eta]$. By Lemma 4.1 there exists $\epsilon > 0$ such that for $l = 1, \dots, k$ and all N large enough

$$\frac{1}{N} \log \max_m |(\mathcal{S}_N + R_N^1) \chi_l h_N(m)| \leq -\epsilon. \quad (4.20)$$

Let \mathcal{S}_N^l be a Dirichlet restriction of \mathcal{S} to $I_\delta(m_l)$. Let $-R_{N,l}^1$ be the leading eigenvalue of \mathcal{S}_N^l .

We are entitled to conclude: There exists $\epsilon' > 0$ such that

$$\frac{1}{N} \log |R_N^1 - \min_l R_{N,l}^1| \leq -\epsilon'. \quad (4.21)$$

Furthermore,

Proposition 4.2. *If $l = 0, \dots, k$ and $\psi = \lim_{j \rightarrow \infty} \psi_{N_j}$ is a subsequential limit such that m_l is a local minimum of ψ , then there exists $\epsilon' > 0$ such that:*

$$\frac{1}{N_j} \log |R_{N_j}^1 - R_{N_j,l}^1| \leq -\epsilon'. \quad (4.22)$$

Proof. In view of Lemma 2.4 the claim readily follows from the general theory of exponentially low lying spectra for metastable Markov chains [6]. For a direct proof note that under the assumptions of the Proposition, one (possibly after further shrinking the value of δ) can upgrade (4.20) as

$$\frac{1}{N_j} \log \max_m \left| \left(\mathcal{S}_{N_j} + R_{N_j}^1 \right) \frac{\chi_l h_{N_j}(m)}{h_{N_j}(m_l)} \right| \leq -\epsilon, \quad (4.23)$$

and (4.22) follows from the spectral theorem. \square

Asymptotics of Dirichlet Eigenvalues $R_{N,l}^1$. Define $\lambda(m) = \sqrt{1 - m^2}$. The asymptotics of $R_{N,l}^1$ up to zero order terms is given [25] by

$$-R_{N,l}^1 = -Nr_1 - \sqrt{\frac{V''(m_l)}{\lambda(m_l)}} + \mathcal{O}\left(\frac{1}{N}\right) = -Nr_1 - \chi_0(m_l) + \mathcal{O}\left(\frac{1}{N}\right), \quad (4.24)$$

where we used the explicit expression (4.4) for V in the second equality. χ_0 was defined in (4.16). The claim of Corollary 4.1 follows now from Proposition 4.2. \square

APPENDIX A. THE VARIATIONAL PROBLEM

The Lagrangian \mathcal{L}_0 was defined in (1.11)

Lower bounds on \mathcal{L}_0 . Fix $\alpha \in \mathcal{A}$ and consider $\theta_\alpha^t = \frac{n-1}{n}t$ and, for $\beta \neq \alpha$, $\theta_\beta^t = -\frac{1}{n}t$. Recall that $\underline{v} \in \mathbb{R}_0^n$, that is $v_\alpha = -\sum_{\beta \neq \alpha} v_\beta$. Therefore, for any α

$$\mathcal{L}_0(\underline{m}, \underline{v}) \geq \sup_t \left\{ tv_\alpha - \sum_{\beta \neq \alpha} \sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta} (\cosh(t) - 1) \right\}$$

Define $\lambda_\alpha(\underline{m}) = \sum_{\beta \neq \alpha} \sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta}$. For $|v_\alpha| \geq \lambda_\alpha(\underline{m})$ one may choose $t^* = \text{sign}(v_\alpha) \log \frac{|v_\alpha|}{\lambda_\alpha(\underline{m})}$. We infer: If $|v_\alpha| \geq \lambda_\alpha(\underline{m})$, then

$$\mathcal{L}_0(\underline{m}, \underline{v}) \geq |v_\alpha| \left(\log \frac{|v_\alpha|}{\lambda_\alpha(\underline{m})} - 1 \right). \quad (\text{A.1})$$

Upper bounds on the Lagrangian \mathcal{L}_0 . Consider

$$\mathcal{R}_0(\underline{m}, \underline{v}) \triangleq \sup_{\underline{\theta}} \left\{ \sum v_\alpha \theta_\alpha - \sum_{\alpha, \beta} \sqrt{m_\alpha m_\beta} \lambda_{\alpha, \beta} \cosh(\theta_\beta - \theta_\alpha) \right\}.$$

Since $\mathcal{L}_0(\underline{m}, \underline{v}) = \mathcal{R}_0(\underline{m}, \underline{v}) + \sum_\alpha \lambda_\alpha(\underline{m})$, it would be enough to control the dependence of \mathcal{R}_0 on \underline{v} .

Let us say that a flow $\underline{f} = \{f_{\alpha\beta}\}$ is compatible with $\underline{v} \in \mathbb{R}_0^n$; $\underline{f} \sim \underline{v}$ if:

- (a) It is a flow: $f_{\alpha\beta} = -f_{\beta\alpha}$.
- (b) For any $\alpha \in \mathcal{A}$, $\sum_\beta f_{\beta\alpha} = v_\alpha$.

Then $\sum v_\alpha \theta_\alpha = \frac{1}{2} \sum_{\alpha, \beta} (\theta_\beta - \theta_\alpha) f_{\alpha\beta}$. Hence, for any $\underline{f} \sim \underline{v}$,

$$\mathcal{R}_0 = \sup_{\underline{\theta}} \left\{ \frac{1}{2} \sum_{\alpha, \beta} (\theta_\beta - \theta_\alpha) f_{\alpha\beta} - \sum_{\alpha, \beta} \sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta} \cosh(\theta_\beta - \theta_\alpha) \right\}. \quad (\text{A.2})$$

We shall rely on the following upper bound on each term in (A.2): For any f and $a > 0$

$$\sup_t \{ft - a \cosh(t)\} \leq |f| \log \left(1 + \frac{2|f|}{a} \right).$$

Consequently, we derive the following upper bound on \mathcal{R}_0 :

$$\mathcal{R}_0(\underline{m}, \underline{v}) \leq \inf_{\underline{f} \sim \underline{v}} \sum_{\alpha, \beta} \frac{|f_{\alpha\beta}|}{2} \log \left(1 + \frac{|f_{\alpha\beta}|}{\sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta}} \right). \quad (\text{A.3})$$

Regularity of minimizers. Let $\underline{m} \in \text{int}(\Delta_d)$. We claim that there exists $\delta_0 > 0$ and $t_0 > 0$ such that for any \underline{m}' in the δ_0 -neighbourhood of \underline{m} the minimizer γ^* of

$$\inf_{\gamma(0)=\underline{m}', \gamma(t_0)=\underline{m}} \int_0^{t_0} \mathcal{L}(\gamma(s), \gamma'(s)) ds$$

exists and is, actually, C^∞ . Indeed, an absolutely continuous minimizer exists by the classical Tonelli's theorem. By lower (A.1) and upper (A.3) bounds on the

Lagrangian, it is easy to understand that minimizers stay inside $\text{int}(\Delta_d)$ once t_0 and δ_0 are chosen to be appropriately small. But then the regularity theory of either [11] or [4] applies and yields Lipschitz regularity on $[0, t_0]$. Since, the Lagrangian \mathcal{L} is strictly convex in the second argument, and, in the interior of Δ_d , it is C^∞ in both arguments, the C^∞ of the minimizer follows from the implicit function theorem, see e.g. [7].

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FACULTY OF INDUSTRIAL ENGINEERING, TECHNION, HAIFA 3200, ISRAEL

E-mail address: `ieioffe@ie.technion.ac.il`

DEPARTMENT OF MATHEMATICS, UBC, VANCOUVER, B.C. V6T 1Z2, CANADA

E-mail address: `anna.levit@math.ubc.ca`